Circulation, Vorticity, and Potential Vorticity

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1. Introduction

Vorticity and circulation are closely related quantities that describe rotational motion in fluids. Vorticity describes the rotation at each point; circulation, over a region. Both quantities are of fundamental importance to the field of fluid dynamics. The distribution and statistical properties of the vorticity field provide a succinct characterization of fluid flow, particularly for weakly compressible or incompressible fluids. In addition, stresses acting on the fluid are often interpreted in terms of the generation, transport, and dissipation of vorticity and the resulting impact on the circulation.

Potential vorticity (PV) is a generalized vorticity that combines information about both rotational motion and the distribution of density. PV is of central importance to the field of geophysical fluid dynamics (GFD) and its sub-fields of dynamic meteorology and physical oceanography. Work in GFD often focuses on flows that are characterized by rapid planetary rotation and strong density stratification by the gravitational field. When so characterized, a fluid flow can be succinctly described by its distribution of PV. Similarly to vorticity, the generation, transport and dissipation of PV is closely associated with stresses on the fluid.

Vorticity, circulation, and PV are described extensively in several textbooks (e.g., Holton 1992, Gill 1982, Kundu 1990, and Salmon 1998). This review is a tutorial, with illustrative examples, that is meant to acquaint the lay reader with these concepts and the scope of their application. An appendix provides a mathematical summary.

2. Circulation and Vorticity: Definitions and Examples

"Circulation" is a physical quantity that describes the net movement of fluid along a chosen "circuit," that is, a directed path that starts at some point and returns to that point.

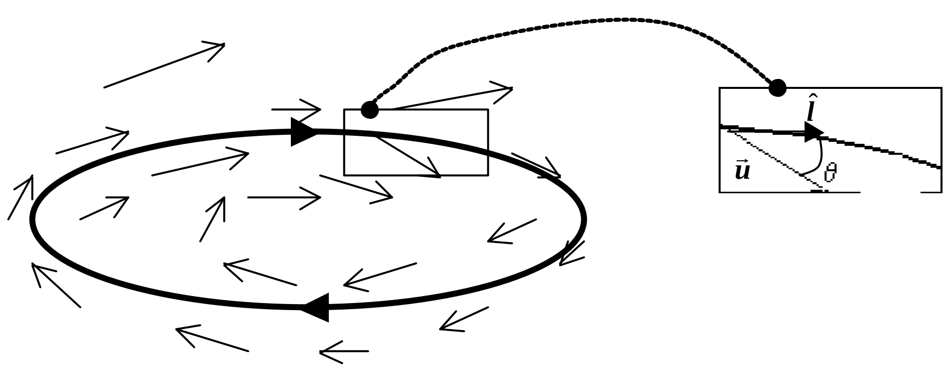


Figure 1: Heavy curve, dark arrows: circuit. Light arrows: flow velocity.

*Definition (1): The **circulation** at a given time is the average, over a circuit, of the component of the flow velocity tangential to the circuit, multiplied by the length of the circuit. Circulation therefore has dimensions of length squared per unit time.*

Consider the circuit drawn as a heavy curve in Fig. 1, for a fluid whose velocity $\vec{\boldsymbol{u}}$ is indicated by the arrows. At each point we may split the velocity into components tangential to and perpendicular to the local direction of the circuit. The tangential component is defined as the dot product $\vec{\boldsymbol{u}} \cdot \hat{\boldsymbol{l}} = |\vec{\boldsymbol{u}}| \cos q$, where $\hat{\boldsymbol{l}}$ is a unit vector that points in the direction tangential to the circuit and q is the angle between $\vec{\boldsymbol{u}}$ and $\hat{\boldsymbol{l}}$. Where q is acute, the tangential component is positive, where oblique, negative, and where a right angle, zero. To calculate the circulation, we average over "all" points along the circuit. Although we cannot actually average over "all" points along the circuit, we can approximate such an average by summing over the tangential component $\vec{\boldsymbol{u}} \cdot \hat{\boldsymbol{l}} = |\vec{\boldsymbol{u}}| \cos q$ at evenly and closely spaced points around the circuit, and divide by the number of points. The circulation would then be this average, multiplied by the length of the circuit. (In the Appendix, the circulation is defined in terms of a line integral.)

The circulation is not defined with reference to a particular point in space but to a circuit that must be chosen. For example, to measure the strength of the primary surface flow around a hurricane in the Northern Hemisphere, a natural circuit is a circle, oriented horizontally, centered

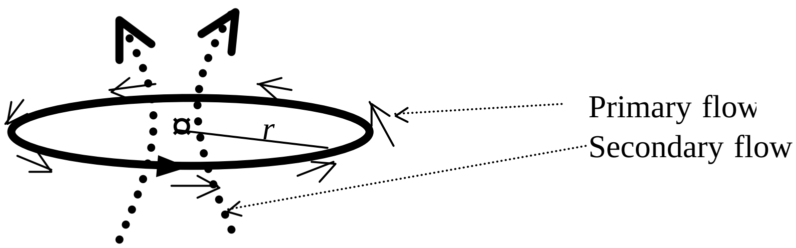


Figure 2: Circuit shown with bold curve. ¤ symbol represents paddle wheel with axis vertical and located at center of circle.

about the hurricane's eye, of radius somewhat larger than the eye, and directed counter-clockwise when viewed from above. This is the cyclonic direction, which is the direction of the primary flow around tropical storms and other circulations with low-pressure centers in the Northern Hemisphere (Fig. 2).

Suppose that the average flow tangential to the circuit is approximately $u = 40\,\mathrm{ms}^{-1}$ and that the radius of the circle is $r = 100$ km. The circulation, using the definition above, is then the average tangential flow times the circumference of the circle: $\Gamma = 2pru \approx 2.5 \text{ x } 10^6 \mathrm{m}^2\mathrm{s}^{-1}$. If the circuit were chosen to run clockwise, i.e. opposite to the primary flow, instead of counter clockwise, the circulation would be of the same strength but of opposite sign.

Circulation is a selective measure of the strength of the flow: it tells us nothing about components of motion perpendicular to the circuit or components of the flow that average to zero around the circuit. For example, suppose the hurricane is blown to the west with a constant easterly wind. This constant wind will not contribute to the circulation: since $\cos(q + p) = -\cos q$, every point on the north side of the circle that contributes some amount to the circulation has a counterpart on the south side that contributes an equal and opposite amount. As another example, the circuit in Fig. 2 would not measure the strength of the hurricane's secondary flow, which runs toward the eye near the surface and away from the eye aloft (dashed arrows in Fig. 2). The reader might try to imagine a circuit that would measure this secondary flow.

The hurricane example suggests that the fluid must flow around the chosen circuit for the circulation to be non-zero; however, this need not be true. For example, consider the circuit shown in Fig. 3, which shows a vertically oriented 50m x 200m rectangular circuit in a west-to-east flow whose strength increases linearly with height, according to the formula $u(z) = az$, where z is the height in meters and $a = 0.01\,\mathrm{s}^{-1}$=10 ms^{-1}/km is the vertical shear. This is a value of vertical shear that might be encountered in the atmospheric boundary layer. If the direction of the circuit is chosen, as in the figure, to be clockwise, the circulation in this example is the average of the along-circuit component of the flow weighted by the length of each side, $\frac{50\times u(\mathrm{top}) - 50\times u(\mathrm{bottom}) + 200\times 0 + 200\times 0}{50+50+200+200} = \frac{50\times 200a}{500} = 0.2\,\mathrm{ms}^{-1}$, times the perimeter of the

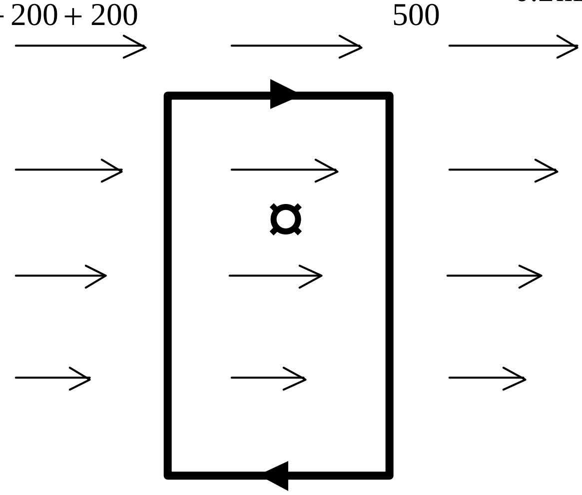


Figure 3: Circuit shown with bold line. ¤ symbol represents paddle wheel, with rotation axis coming out of page.

rectangle, giving Γ=100m^{2}s^{-1}. Note that the vertical sides make no contribution to the circulation since the flow is perpendicular to them. Thus, shear alone can make the circulation non-zero even though the flow is not actually moving around the circuit.

What, then, does the circulation represent in these examples, if not the strength of the flow around the circuit? In general, circulation represents the ability of a given flow to rotate mass elements in the fluid and objects placed in the flow. Imagine a paddle wheel placed within the circuits in Figs. 2 and 3 (see e.g. Holton 1992, Fig. 4.6), with axis perpendicular to the plane of the circuit. In the hurricane illustration, the paddle wheel rotates counter-clockwise; in the vertical shear-flow, clockwise. In both examples, when the paddle wheel turns in the same sense as the circuit in the illustration, the sign of the circulation is positive.

Circulation depends on many details about the circuit: its size, shape, orientation, sense, and location. For example, in Fig. 3, we could make the circulation arbitrarily large by increasing the length of the sides of the circuit. It is useful to define a quantity that measures rotation but that does not refer to the details of any particular circuit. It is also useful to define a "microscoic" quantity that measures rotation at individual points instead of a "macroscopic" quantity that measures rotation over a region. This leads us to “vorticity.”

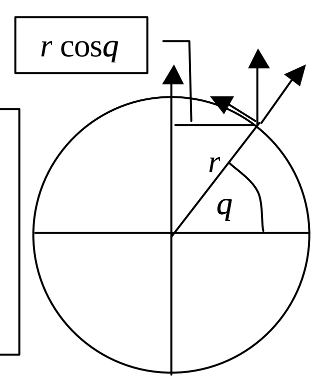
> *Definition (2): The **vorticity** in the right-hand-oriented direction normal to the plane of a circuit is equal to the circulation per unit area of the circuit in the small-area limit. Vorticity is, therefore, a quantity with dimensions of inverse time. In order to define the x, y, and z components of the vorticity, we consider circuits whose normal lies in each of these directions.*

Definition (2) requires some explanation. Consider the circulation for a circuit, which, for simplicity, we assume to be flat. This defines a unique direction that is perpendicular to the plane of the circuit if we use the “right-hand rule.” This rule works as follows: curl the fingers of the right hand in the direction of the circuit, e.g. clockwise in Fig. 3. Then point the thumb in a direction perpendicular to the circuit: into the page in Fig. 3. In definition (2), we refer to this direction as the “right-hand-oriented direction normal to the plane of the circuit.” Now, imagine reducing the size of the circuit until it becomes vanishingly small. In Fig. 3, for example, we could imagine reducing the loop to 5m x 20m, then 5mm x 20mm, and so on. In definition (2), we refer to this as the “small-area limit.” Therefore, vorticity describes rotation as a microscopic quantity.

We can calculate vorticity in the previous examples. For the shear-flow example (Fig. 3), it is easy to show that the circulation around the loop is aA, where A is the area of the rectangular loop. With more effort, it can be shown that the circulation is aA for any loop if A is taken to be the cross-sectional area of the loop in the plane of the flow. Therefore, the vorticity in the plane of the flow is into the page and equal to $a = 0.01\text{s}^{-1}$ everywhere. The reader should try to verify that the other two components of the vorticity are zero because the accompanying circulation is zero. In hurricanes, as is typical of other vortices, the vorticity varies strongly as a function of distance from the eye. The average vertical component of vorticity in our example is upward and equal to the circulation divided by the total area of the circle: $2pru/(pr^2) = 2u/r = 10^{-3}\text{s}^{-1}$.

In GFD, there are three types of vorticity: absolute, planetary, and relative. The **absolute vorticity** is the vorticity measured from an inertial frame of reference, typically thought of as outer space. The **planetary vorticity** is the vorticity associated with planetary rotation. Earth rotates with a period of $T \approx 24$ hours from west to east around the North-Pole South-Pole rotation axis (Fig. 4). Consider a fluid with zero ground speed, meaning that it is at rest when measured from the frame of reference of the earth. This fluid will also be rotating with period T from west to east. The fluid is said to be “in solid-body rotation,” because all the fluid elements maintain the same distance from one another over time, just as the mass elements do in a rotating solid. The distance from the axis of rotation is $r\cos q$, the component of the velocity tangential to this rotation is $2pr\cos q / T$, the circumference of the latitude circle is $2pr\cos q$, and the circulation, by definition (1), is the product

Figure 4: Geometry of Earth. Latitude is θ. Vertical vector is vorticity of solid body rotation. Distance from surface to axis of rotation is $r\cos q$. Component of planetary vorticity perpendicular to surface is $4p\sin q / T$. Component tangential to surface is $4p\cos q / T$.



of these two quantities $(2\pi r\cos\theta)^2/T$. The vorticity points from the South Pole to the North Pole and has magnitude $4\pi/T = 1.4\times10^{-4}\text{s}^{-1}$, where we have divided the circulation by the area of the latitude circle. This is the planetary vorticity. Finally, the **relative vorticity** is the vorticity measured using velocities measured with respect to the ground, in the rotating frame of reference of the Earth. The absolute vorticity is the sum of the relative vorticity and the planetary vorticity.

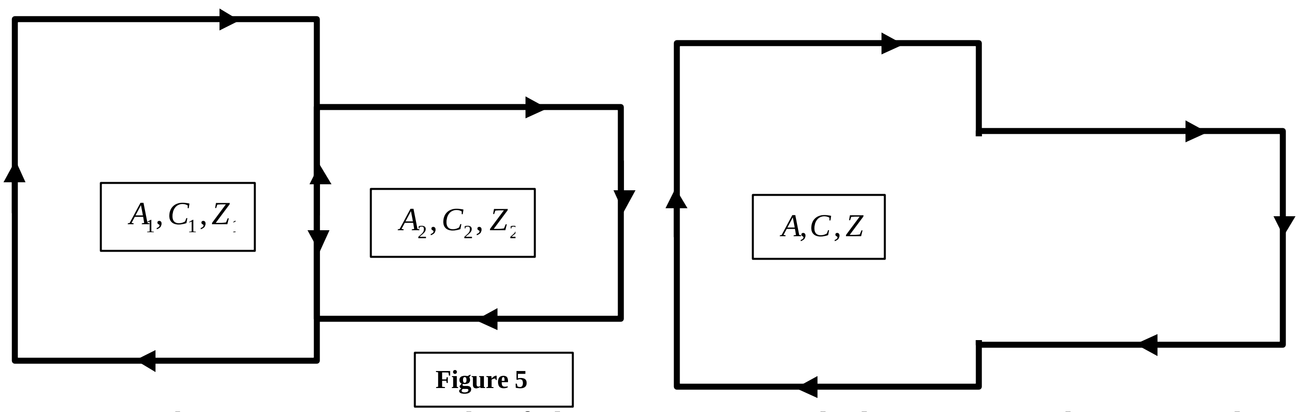


Figure 5

The comparative strengths of planetary vorticity and relative vorticity determine, in large part, the different scaling regimes of GFD. The planetary vorticity has a component perpendicular to the planet's surface with value $4\pi\sin\theta/T$. This component points radially, away from the center of the planet in the Northern Hemisphere where $\sin\theta > 0$, and toward the center of the planet in the Southern Hemisphere where $\sin\theta < 0$. From the viewpoint of an observer on the surface in the Northern Hemisphere, this component points vertically up toward outer space and, in the Southern Hemisphere, vertically down toward the ground. There is also a tangential, northward component of the planetary vorticity, $4\pi\cos\theta/T$. For motions that are characterized by scales of 1000 km or greater in the atmospheric midlatitudes, or 100 km or larger in the oceanic midlatitudes, the radial, "up/down" component of the planetary vorticity is an order of magnitude larger than the relative vorticity in this direction. This is the scaling regime for midlatitude cyclones and oceanic mesoscale eddies, for which geostrophic balance and quasi-geostrophy hold (see Chapter ??). At scales smaller than this, the relative vorticity can be comparable to or larger than the planetary vorticity. For instance, in the hurricane example, the vertical component of the vorticity was determined to be 10^{-3}s^{-1} in a region 100 km across. Often even larger are the values of the horizontal components of vorticity associated with vertical shear -- recall the vertical shear example, with vorticity of magnitude 10^{-2}s^{-1}. Although it is large values of the vertical component of vorticity that are associated with strong horizontal surface winds, the availability of large values of horizontal vorticity associated with vertical shear can have a potentially devastating impact. For example, the tilting into the vertical of horizontal shear vorticity characterizes the development of thunderstorms and tornadoes (e.g. Cotton and Anthes, 1989).

Having defined vorticity in terms of the circulation, it is also useful to go from the microscopic to the macroscopic and define circulation in terms of the vorticity. We first introduce the idea of a vector **flux**: the flux of any vector field through a surface is the average of the component of the vector normal to the surface, multiplied by the area of the surface. For example, the flux of a 10 ms^{-1} flow passing through a pipe of cross sectional area 0.5 m^2 is a volume flux of 5 m^3s^{-1}.

Statement (3): The circulation for a given circuit is equal to the flux of the vorticity through any surface bounded by the circuit.

To illustrate statement (3), we consider Fig. 5, which shows adjacent rectangular circuits with circulation values C_1, C_2 and areas A_1, A_2 that are small enough to have unique values of the vorticity $Z_1 = C_1 / A_1$, $Z_2 = C_2 / A_2$ pointing into the page. The total circulation for the larger rectangular region formed by joining the two smaller rectangles is $C = C_1 + C_2$. This is because, along the shared side (marked in the figure with circulation arrows pointing in both directions), the tangential flow component for circuit 1 is equal and opposite to that for circuit 2. The average component of the vorticity for this region is $(Z_1 A_1 + Z_2 A_2)/(A_1 + A_2) = (C_1 + C_2)/(A_1 + A_2) = C / A$, where A is the total area. The flux of vorticity is then $(C / A) \times A = C$, which is the total circulation, consistent with statement (3). We can repeat this calculation to include more rectangles in order to determine the average component of the vorticity over a large region. Although we have illustrated the simple case in which the surface bounded by the circuit is flat, statement (3) holds for any surface, flat or curved, bounded by the circuit. This is because (3) is a statement of Stokes theorem of vector calculus (see Appendix).

Statement (3) shows that vorticity distributed over a small region can be associated with a circulation well away from that region. Typical vortices in geophysical flows tend to have a core of strong vorticity surrounded by a core of relatively weak or zero vorticity. Suppose the vortex covers an area A and has a perpendicular component of vorticity of average value Z. The circulation induced by this flux, for any circuit enclosing the vortex, is AZ. Consider a circuit that spans an area larger than A. The perimeter of such a circuit will scale as its average distance l from the center of the vortex. Recall the hurricane example, for which the radius is l and the perimeter is $2pl$. Then the induced tangential flow speed associated with the vortex, from definition (1), scales as AZ / l. That is, for typical localized vortex distributions, the flow around the vortex scales with the reciprocal of distance from the center. The ability of vorticity to induce circulation "at a distance" and the scaling with the reciprocal of distance of the flow strength are key to understanding many problems in GFD.

Statement (3) also implies that mechanisms that change the vorticity in some region can change the circulation of any circuit that encloses that region. For instance, in the hurricane example of Fig. 2, suppose that drag effects near the surface reduce the vertical component of the vorticity in some location (see next section). By (3), this would reduce the average circulation around the circuit. In other words, the reduction in vorticity would decelerate the flow around the circuit. In this way, we see that vorticity transport, generation, and dissipation give rise to stresses that can accelerate or decelerate the flow.

In the final example of this section, we will illustrate, with an idealized model, how vorticity transport gives rise to flow accelerations on a planetary scale (I. Held, personal communication). A thin layer of fluid of constant density and depth surrounds a solid, featureless, “billiard ball” planet of radius r. By "thin," we mean that the depth of the fluid is much less than r. Both planet and fluid layer are rotating, with period T, from west to east. As we have seen, the vorticity of the solid body rotation is the planetary vorticity, and the component of this vorticity perpendicular to the surface is $4p \sin q / T$. We will learn, shortly, that this normal component, in the absence of applied forcing or friction, acts like a label on fluid parcels and stays with them as they move around. In other words, the component of the vorticity normal to the surface of the fluid is a tracer.

Suppose, now, that a wavemaker near the equator generates a wave disturbance, and that this wave disturbance propagates away from the region of the wavemaker. Away from the wavemaker, the parcels will be displaced by the propagating disturbance. Given that the component of the vorticity $4p \sin q / T$ normal to the surface is a tracer, fluid elements so displaced will carry their initial value of this quantity with them. For example, a fluid element at 45N latitude will preserve its value of vorticity, $4p \sin(p/4)/T = 1.0 \times 10^{-4}\,\mathrm{s}^{-1}$ as it moves north and south. Now, $4p \sin q / T$ is an increasing function of latitude: there is a gradient in this quantity, from south to north. Therefore, particles from low vorticity regions to the south will move to regions of high vorticity to the north, and vice versa. There will be, therefore, a net southward transport of vorticity, and a reduction in the total vorticity poleward of that latitude. Thus, the circulation around that latitude will be reduced: the fluid at that latitude will begin to stream westward as a result of the disturbance. The transport of vorticity by the propagating disturbance gives rise to a stress that induces acceleration on the flow. This acceleration is in the direction perpendicular to the vorticity transport.

We can estimate the size of the disturbance-induced acceleration. Suppose, after a few days, that particles are displaced, on average, by one degree latitude, which corresponds to a distance of about 100 km. Since $4p \sin q / T$ has values between $-1.4 \times 10^{-4}\,\mathrm{s}^{-1}$ and $1.4 \times 10^{-4}\,\mathrm{s}^{-1}$ for $T = 24\,\mathrm{h}$, a reasonable estimate for the difference between a displaced particle's vorticity and the background vorticity is $10^{-5}\,\mathrm{s}^{-1}$ for a one-degree latitude displacement. The average estimated southward transport of the vorticity is then the displacement times the perturbation vorticity: $100\,\mathrm{km} \times 10^{-5}\,\mathrm{s}^{-1} = 1\,\mathrm{ms}^{-1}$. This is an estimate of the westward flow velocity induced by the displacement over a few days, and corresponds to reasonable values of the observed eddy-induced stresses on the large-scale flow.

3. Potential Vorticity: Definition and Examples

The previous example describes the effect on the horizontal circulation of redistributing the vorticity of solid-body rotation. Although the example seems highly idealized, the wave-induced stress mechanism it illustrates is fundamental to the Earth's large-scale atmospheric circulation. The physical model of a thin, fixed-density, and constant-depth fluid, which is known as the barotropic vorticity model, is deceptively simple, but has been used as a starting point for an extensive body of work in GFD. This work ranges from studies of the large-scale ocean circulation (Pedlosky 1996), to the analysis of the impact of tropical disturbances such as El Niño on the midlatitude circulation, and to early efforts in numerical weather forecasting. Such applications are possible because the idealizations in the example are not as drastic as they initially appear. For example, the earth's atmosphere and ocean are quite thin compared to the radius of the Earth: most of the atmosphere and world ocean are contained within a layer about 20 km thick, which is small compared to the Earth's radius (about 6300 km). In addition, large-scale atmospheric speeds are characteristically 10-20 m/s, which is small compared to the speed of Earth's rotation ($2pr \cos q / T = 460 \cos q$ $\mathrm{ms}^{-1} \approx 300\ \mathrm{ms}^{-1}$ in the midlatitudes) — the atmosphere is not so far from solid-body rotation. This is consistent with the idea that at large scales, atmospheric and oceanic flows have vertical vorticity components that are much smaller than the planetary vorticity. Perhaps the most drastic simplifications in the example are that the layer of fluid has constant depth and constant density. In this section, we consider variations of fluid layer depth of fluid density; this will lead us to potential vorticity (PV).

Since large-scale atmospheric circulations typically occur in thin layers, these fluids are typically “hydrostatic.” This means that there is a balance between the vertical pressure force and the force of gravity on each parcel, that vertical accelerations are weak, and that the pressure at any point is equal to the weight per unit area of the fluid column above that point. Consider a thin hydrostatic fluid of constant density but of variable depth. Scaling arguments show that the state of the fluid may be completely specified by three variables: the two horizontal components of the velocity, and the depth of the fluid. The system formed by these three variables and the equations that govern them is known as the **shallow-water model**. These three variables are functions of longitude and latitude alone, which implies that there is only a single component of vorticity: the vertical component of vorticity associated with the north-south and east-west components of motion.

The shallow-water model provides a relatively simple context to begin to think about PV:

*Definition (4): The **shallow-water PV** is equal to the absolute vorticity divided by the depth of the fluid; it has dimensions of inverse time-length.*

By this definition, the PV can be changed by either changing the vorticity or by changing the depth of the fluid column. The depth of the fluid column can be changed, in turn, by the fluid column encountering variations in depth of the topography below the fluid or by varying the height of the fluid’s surface. For example, if the PV of a fluid column is held constant and the depth of the fluid decreased, by, for example, shoaling (i.e. moving the column up a topographic slope), the result is a reduction in the column’s vorticity.

The generalization of PV to fluids in which the density is not constant can be approached in stages. To start with, consider, instead of a single layer of fluid, a system consisting of two or more thin, constant-density, hydrostatic layers, in which each layer lies under another layer of lighter fluid. For this system, the PV in each layer is the ratio of the vertical vorticity to the layer depth. If density is instead taken to vary continuously, extra complications are added. First, at least six variables are required to specify the state of the fluid: the three components of the velocity, the pressure, the density, and another thermodynamic variable such as the temperature or the specific entropy. The specific entropy is the entropy per unit mass (see Sec. 4). Additional variables will be needed to account for salinity in the ocean and moisture in the atmosphere; we neglect these additional factors. The second complication is that, since the velocity is three dimensional, so, now, is the vorticity vector. The generalization of the shallow-water PV to fluids with variable density is known as "Ertel’s PV."

*Definition (5): **Ertel's PV** is equal to the projection of the absolute vorticity vector onto the spatial gradient of the specific entropy, divided by density. Its dimensions depend on the physical dimensions of the entropy measure used.*

To explain definition (5) in more detail: recall that the projection of vector $\vec{\boldsymbol{a}}$ onto vector $\vec{\boldsymbol{b}}$ is $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}}$. The gradient of the specific entropy is a vector, pointing perpendicular to a surface of constant specific entropy, in the direction of increasing specific entropy. Its value is equal to the rate of change of specific entropy per unit along-gradient distance (Fig. 6). Despite the additional complexity, there are analogies between definition (4) and definition (5). For example, for fixed PV, we can decrease a column's vorticity by decreasing the thickness between two surfaces of specific entropy, since this would decrease the gradient. This reduction in thickness is analogous to compressing the fluid column, as in the shallow-water example. The connection between definitions (4) and (5) will be discussed in more detail below.

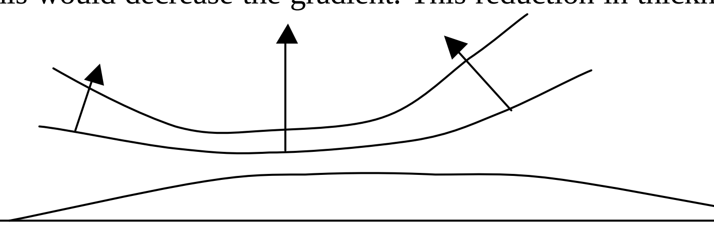


Figure 6: Contours: surfaces of constant specific entropy. Arrows: gradient, pointing in direction of increasing specific entropy.

We have defined circulation, vorticity, and PV, but have made little explicit reference to fluid dynamics, that is, to the interactions and balances within a fluid that allow us to predict its behavior. Dynamical considerations justify our interest in defining PV as we have here, and link the PV back to our original discussion of circulation. Another important loose end is to look at the impact of planetary rotation in more detail, since rotation dominates large-scale motions on Earth. These topics will be taken up in the next section.

4. Dynamical Considerations: Kelvin's Circulation Theorem, Rotation, and PV Dynamics

We begin our discussion of the dynamical aspects of circulation, vorticity, and PV by returning to the shallow water system, which consists of a thin hydrostatic layer of fluid of constant density. In this section, we will need to introduce several new concepts. First, we introduce the concept of a **material circuit**. Points on a material circuit do not stay fixed in space, as in the circuits in Figs. 2 and 3, but instead follow the motion of the fluid. In the presence of shear, mixing, and turbulence, material circuits become considerably distorted over time.

Statement (6): In the absence of friction and applied stresses, the absolute circulation in the shallow-water model is constant for a material circuit.

Statement (6) asserts that the circulation measured along a material circuit, as it follows the flow, will not change. This is a special case of Kelvin's circulation theorem, which applies to a fluid with variable density, and which will be discussed shortly.

The circulation theorem (6) follows from applying, to a material circuit, the fluid momentum equations that express Newton's second law of motion. Newton's second law expresses the balance between the acceleration of the flow and the sum of the forces per unit mass acting on the fluid. For the shallow-water fluid, these forces include the pressure force, friction near solid boundaries, and applied stresses (such as the stress exerted by the wind on the ocean). The pressure force is not mentioned in statement (6) because it cannot generate circulation on any circuit in the shallow-water system. To understand this, we need to remind the reader of the concept of **torque**. Torque is a force that acts on a body through a point other than the body's center of mass. Forces that act through a body's center of mass cause acceleration in the direction of the force. Torque, on the other hand, causes rotation. For example, it is a torque imparted by the vorticity in the fluid that sets spinning the paddle wheels in Figs. 2 and 3. Torque is a vector, which, by convention, points in the direction of the imparted rotation using the right-hand rule.

The pressure force in the shallow-water system acts through the center of fluid columns. Therefore, the pressure force cannot directly impart a rotational torque to fluid elements. Nevertheless, pressure forces can compress or expand the circuit horizontally. Variations in the area of the circuit then induce rotation indirectly, because of the circulation theorem (6). To see this, suppose a material circuit that surrounds a fluid column is compressed horizontally. We note that the mass of the column is another quantity that is constant following the motion – in the absence of mass sources and sinks, no mass will leak into or out of the vertical column that the circuit encloses. Since the density is a constant throughout the fluid, and since the mass is constant following the motion, the volume of the column must be conserved following the motion of the fluid. Therefore, horizontal compression increases proportionally the height of the column. At the same time, if the area of the material circuit is reduced, (6) shows that the vorticity within the circuit must increase to maintain a constant circulation. The increase in vorticity associated with horizontal compression and vertical stretching is a statement of local angular momentum conservation (see e.g. Salmon 1998).

As pointed out in the previous section, the depth of the fluid column may also change if the column passes over topographic features of the solid underlying surface. Columns passing over ridges obtain negative circulation, over valleys, positive circulation.

The interpretation of (6) in the presence of planetary rotation is fundamental to the study of large-scale GFD. Statement (6) holds for the absolute circulation, that is, the circulation measured from an inertial, non-rotating frame. An important mechanism for generating relative vorticity involves the constraint of having to conserve total circulation as fluid moves North and South. Consider a ring of fluid, at rest in the rotating frame, of small surface area A, and located at latitude θ. From our earlier discussion of the rotating fluid on the sphere, the vorticity for this ring is the planetary vorticity $4\pi \sin\theta / T$ and the circulation for this ring is the planetary circulation $4\pi A \sin\theta / T$. If the fluid is free of friction and external applied stresses, then (6) tells us that the sum of the planetary and relative circulations will be constant following the motion of the ring. Thus, if the ring maintains its original area and is displaced northward, it will acquire a positive (cyclonic, counter clockwise in the Northern Hemisphere) relative circulation. In this way, the south-to-north planetary-vorticity gradient provides an important constraint on the motion of the large-scale flow.

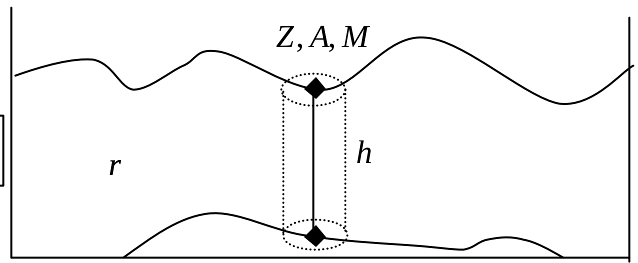
The final application of (6) we will consider concerns the effect of applied stresses and friction on the circulation. The proof of theorem (6), which we have not detailed, shows how circulation may be generated under conditions that depart from the theorem's assumptions. In particular, frictional and applied stresses, such as those found in atmospheric and oceanic boundary layers, can impart vorticity to a fluid by a mechanism known as "Ekman pumping." Ekman pumping describes the way a fluid boundary layer responds to interior circulation. For example, a positive (cyclonic) relative circulation can cause upwelling in the bottom boundary layer, which tends to decrease the depth of the fluid column and therefore counteract the positive circulation. On the other hand, applied stresses (such as atmospheric wind stress on the ocean) play a major role in generating large-scale oceanic circulation. These effects can be modeled in a simple way within the shallow-water system.

Just as the macroscopic circulation has a microscopic counterpart, vorticity, the macroscopic statement of conservation of circulation (6) has a microscopic counterpart.

> *Statement (7): In the absence of friction and applied stresses, the shallow-water PV (4) is constant following the motion of the fluid.*

To demonstrate (7), we apply the following argument, using the circulation theorem (6) in the

Figure 7



small-area limit (see Fig. 7):

a) In the small-area limit, the circulation ZA is a constant of the motion, where Z is the vorticity and A is the area of the material circuit.
b) $ZA = (ZM)/(rh)$, where M is the mass of the fluid enclosed by the circuit, r the density, and h the depth.
c) Recall that M is constant following the motion, and that r is a simple constant. Thus,
d) Z/h, which is the shallow-water PV (4), is also a constant of the motion.

That PV is a constant following the motion is consistent with the example in which the vorticity increased proportionally to, and the height, inversely proportionally to, the horizontal compression of the column. All the important mechanisms discussed in the context of (6) — the roles of fluid column stretching and compression by the pressure force and topography, and the ability of planetary rotation to induce relative circulation as systems move north and south — can be interpreted in terms of the conservation of shallow-water PV (7). The PV can be thought of as the planetary vorticity a fluid column would have if it were brought to a reference latitude and depth. This is the origin of the word "potential" in the name "potential vorticity."

At the end of Section 2, we discussed an example that used the barotropic vorticity model, that is, a thin fluid of constant density and depth. For fixed depth, the circulation theorem (6) is the same, but the PV is now, simply, the absolute vorticity. That is, in the absence of column stretching, the absolute vorticity is constant following the motion. This property was used to argue that particles disturbed by the propagating wave in the example would retain their initial absolute vorticity values.

The final topic of this section concerns the dynamics of fluids with variable density. If density is not constant, it is possible for the pressure force to induce a rotational torque. This will result in an important modification to the circulation theorem. To start with, we define a “barotropic” fluid to be one for which surfaces of constant density are aligned with surfaces of constant pressure. Put in another way, in a barotropic fluid, the density is a function only of the pressure, and not of the temperature or other thermodynamic variables.

> *Statement (8): In a barotropic fluid without friction or applied stresses, the circulation following the motion of the fluid is constant (Kelvin's circulation theorem).*

Similarly to statement (6), statement (8) is proven by applying the momentum equations to a circuit. In a barotropic fluid, the pressure force around a fluid element points through its center of mass, as in the shallow-water system. In this case, the pressure force cannot exert a torque on the fluid. A fluid that is not barotropic is said to be “baroclinic.” In baroclinic fluids, the net pressure force on a fluid element does not pass through the center of mass of the element. This results in a torque on the element that generates rotation, that is, vorticity, and, therefore, circulation. The impact of baroclinicity can be felt at small and large scales in the atmosphere and ocean. On scales of a few kilometers, the baroclinicity associated with the thermal contrast between land and sea generates sea breeze circulations. On planetary scales, it is the baroclinicity associated with the large-scale equator to pole temperature gradient that provides the source of energy and stirring for midlatitude atmospheric and oceanic eddies.

As for the shallow water case, Kelvin's circulation theorem (8) has a microscopic counterpart.

> *Statement (9): In the absence of friction, applied stresses, and applied heating, Ertel’s PV, (5), is constant following the motion of the fluid. This result holds, with additional restrictions, for fluids of variable composition, such as moist air and saline water.*

In order to justify statement (9), we need to discuss the concept of specific entropy in more detail. In thermodynamics, entropy is a thermodynamic variable that can be expressed as a function of other thermodynamic variables, such as temperature, pressure, and density. When heat is applied to a thermodynamic system, the change in entropy is related to the amount of heat transferred to the system. In the absence of heating, the entropy of the system does not change: the system is said to be “isentropic” or “adiabatic.” In an adiabatic fluid, this implies that the entropy per unit mass, i.e. the specific entropy, is a constant of the motion. Therefore, surfaces of constant entropy (isentropic surfaces) are material surfaces, that is, surfaces that move with the fluid. In other words, the motion in an adiabatic fluid is along isentropic surfaces. Adiabatic fluids are relevant to GFD because large-scale atmospheric and oceanic flows are often approximately adiabatic or weakly diabatic.

Now, let us consider a simple fluid such as an ideal gas, for which entropy is a function of pressure and density. Suppose the fluid is adiabatic, so that the flow is along isentropic surfaces. On these surfaces, the pressure is a function of density. That is, the flow along isentropic surfaces in an adiabatic fluid is barotropic. This implies that (7) holds: the circulation is constant following the flow on an isentropic surface. Now, consider (7) applied in the small-area limit to two closely spaced isentropic surfaces (Fig. 8). The demonstration of (9) proceeds as follows:

a) In the small-area limit, the circulation ZA is a constant of the motion, where Z is the component of the vorticity normal to the isentropic surface and A is the area of each circuit in the small-area limit,
b) The entropy difference between the two isentropic surfaces, s, is a constant of the motion, since the entropy value of each surface is a constant of the motion, therefore,
c) The product AZs is a constant of the motion,

d) The area $A = V / l$, where V is the volume enclosed by the material column and l is the perpendicular distance between the isentropic surfaces,
e) The volume $V = M / r$, where r is the density, therefore
f) The product $MZs/(lr)$ is a constant of the motion,
g) The mass enclosed by the fluid column M between the two isentropic surfaces is conserved,
h) The product $(Z / r)(s / l)$ is a constant of the motion.
i) The quantity s / l represents the change of entropy per unit distance between the isentropic surfaces, and therefore represents the specific entropy gradient. Therefore,
j) The quantity $(Z / r)(s / l)$ is Ertel's PV (5) and is a constant of the motion. This demonstrates (9).

The dynamics of Ertel's PV incorporate similar mechanisms to the one discussed for shallow-water PV dynamics, with the fluid depth replaced by the distance between isentropic surfaces. It is very useful to have simpler model systems, such as the shallow water model, with which to build up our intuition concerning PV dynamics. For example, there is a strong analogy between changes

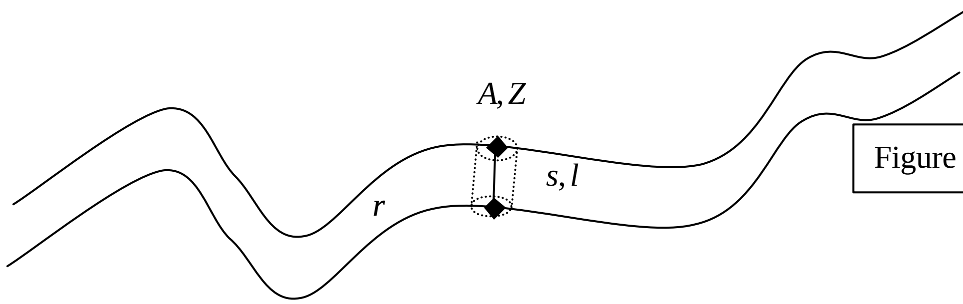


Figure 8

in thickness of the fluid column brought about by mass sources and sinks in the shallow-water model and thermally induced changes in the distribution of isentropic surfaces.

5. Conclusion

PV is a powerful unifying tool in the atmospheric and oceanic sciences, because it combines apparently distinct factors, such as topography, stratification, relative vorticity, planetary vorticity, into a single dynamical quantity. Indeed, much current work in the field is characterized by "PV thinking:" how PV is generated, maintained, converted from one form to another, transported, and dissipated. To conclude, there follows a list of a few important examples of topics of current interest in which PV dynamics is central. Hoskins et al. (1985), Salmon (1998), and Andrews et al. (1987) are excellent starting points for further investigation.

- **Rossby-wave dynamics:** The primary large-scale oscillations in the extratropical atmosphere and oceans are Rossby waves, which are supported by the PV gradients largely associated with planetary rotation. In the "billiard ball" planet example discussed in Section 2, it is Rossby waves that carry the signal of the wavemaker away from the equator. Rossby-wave dynamics underlies our understanding of the midlatitudes remote response to El Nino events in the tropical Pacific, of the propagation of upper-level disturbances, and of the dynamics of large-scale waves in the stratosphere.
- **Baroclinic instability:** The development of atmospheric cyclones and of oceanic eddies can be understood in terms of the interactions of regions of PV gradients with mixed sign.

- **Geostrophic turbulence:** Our understanding of atmosphere/ocean large-scale turbulence is framed by PV, which behaves as an "active" tracer that can be mixed and diffused downgradient.
- **Balance models:** Balance models, such as the quasigeostrophic model (Ch.??), are filtered equations that model the slow, large-scale motions associated with Rossby waves, baroclinic instability, and geostrophic turbulence. These models are based on some of the scaling regimes outlined above, for example, the scaling assumption that the vertical component of relative vorticity is small compared to the vertical component of planetary vorticity. In these models, the PV is the sole dynamical variable, and all other variables can be obtained from the PV by an inversion procedure. This inversion procedure is analogous to the one involved in obtaining the circulation from the vorticity distribution.
- **Rossby wave-activity propagation and wave-induced circulations:** Recall that the southward flux of vorticity in the barotropic vorticity model example at the end of Sec. 2 gives rise to an eastward acceleration. More generally, the flux of PV in both the shallow-water model and fluids with variable density can be associated with a wave-induced torque, generally in the direction transverse to the flux. In this way, the existence of persistent extratropical jets, such as the atmospheric jet stream and the jets in the Antarctic Circumpolar Current, can be understood in terms of the flux, often downgradient, of PV. These fluxes are also the starting point for a theory of "wave-activity" propagation that is the foundation of models of the overturning circulation in the stratosphere.
- **Symmetries, conservation laws, and Hamiltonian structure:** The existence of PV as a constant of the motion has a profound connection to the mathematical structure of the equations of motion. All conserved quantities for a physical system — spatially distributed quantities such as energy and momentum, and local quantities such as PV and wave activity, are connected to symmetries in the system through Noether's theorem. This idea is the starting point for Hamiltonian fluid dynamics, which is reviewed in Salmon (1998). The symmetry connected to the PV is the so-called “particle re-labeling” symmetry, which reflects the fact that the dynamics of an adiabatic fluid is unaffected by changing the particle labels on isentropic surfaces. Such considerations provide a theoretical basis for wave-activity invariants and generalized non-linear stability theorems for GFD, as well as for the systematic derivation of balance models with desirable symmetry.

Appendix: Mathematical Summary

In this section, we provide a summary, using the notation of vector calculus, of the main results. First, the circulation, C, in definition (1), may be represented by a line integral

$$C = \oint_{\text{circuit}} |\vec{\boldsymbol{u}}| \cos\theta \; \mathrm{d}s$$

In this notation, the length of the circuit is $L = \oint_{\text{circuit}} \mathrm{d}s$ and the average of the tangential flow component is C/L. This is consistent with definition (1). By Stokes theorem, the circulation also satisfies

$$C = \iint_{\text{region}} \operatorname{curl}\vec{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}A\,,$$

where the double-integral sign and the word “region” indicate that a flux integral is being taken over the region bounded by the circuit. The vector curl of the velocity is, in Cartesian coordinates,

$$\operatorname{curl}\vec{\mathbf{u}} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

Definition (2) in the small-area limit implies that the vorticity, $\vec{\mathrm{z}}$, is equal to the curl of the velocity:

$$\vec{\mathrm{z}} = \operatorname{curl}\vec{\boldsymbol{u}}\,.$$

Statement (3) is therefore equivalent to Stokes theorem for finite-area surfaces.

For the shallow-water system, the vorticity is vertical:

$$\vec{\mathrm{z}} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{\boldsymbol{z}}$$

and the potential vorticity, definition (4), is

$$q = \frac{\left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} \right)}{h - h_{\mathrm{b}}}\,,$$

where h is the height of the fluid surface and h_{b} is the height of the solid surface beneath the fluid. Ertel’s PV, definition (5), is

$$q = \vec{\mathrm{z}} \cdot \nabla s / r\,,$$

where s is the specific entropy and r is the density.

The circulation theorem for the shallow-water model, statement (6), and Kelvin’s circulation theorem (8), are written

$$DC/Dt \ = \ 0\,,$$

where D/Dt is the material derivative following the motion, defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{\boldsymbol{u}} \cdot \nabla$$

and C is the circulation of a material circuit. The conservation of PV following the motion for shallow-water (7) or three-dimensional stratified flow (9) is

$$Dq/Dt \ = \ 0\,.$$

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